MATH/CSE 1019 Test 1

Fall 2012

Oct 15, 2012

1. (9 points) Propositional Logic. Express these statements using the following propositions.

   - p “The US Central Bank decreases the interest rate.”
   - q “The Bank of Canada increases the interest rate.”
   - r “The US unemployment rate increases."
   - s “The Canada employment rate increases."

   (a) (3 points) The US Central Bank decreases the interest rate, but the US unemployment rate increases.

   $p \wedge r$

   (b) (3 points) If the US Central Bank decreases the interest rate and the Bank of Canada does not increase the interest rate, then the Canada employment rate increases.

   $(p \wedge \neg q) \longrightarrow s$

   (c) (3 points) If the Canada employment rate does not increase, then the Bank of Canada does not increase the interest rate.

   $\neg s \longrightarrow \neg q$

2. (4 points) Propositional equivalences. Show $\neg (p \oplus q)$ and $(p \leftrightarrow q)$ are logically equivalent.

p	q	$p \oplus q$	$\neg (p \oplus q)$	$p \leftrightarrow q$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

$\neg$ (p $\oplus$ q) and p $\leftrightarrow$ q have the same values for all possible p and q value. So they are logically equivalent.

Note: You may also use laws to prove the equivalence as long as the steps are correct.

3. (4 points) Quantifiers. Express the following using predicates, quantifiers and mathematical operators where the domain is all real numbers.

   "The absolute value of the difference of two real numbers does not exceed the sum of the absolute values of these real numbers."

$$\forall \boldsymbol{x} \forall \boldsymbol{y}(|\boldsymbol{x}-\boldsymbol{y}| \leq |\mathbf{x}| + |\boldsymbol{y}|)$$

4. (4 points) Write down the negation of the following statement so that no negation "¬" precedes a quantifier. Then determine whether the statement or its negation is true, and explain why. The domain consists of all real numbers.

$$\forall x \exists y((x>y) \vee (y>x^2))$$

Negation:

$$\begin{aligned}
\neg \forall x \exists y((x>y) \vee (y>x^2)) &\equiv \exists x \neg \exists y ((x>y) \vee (y>x^2)) \\
&\equiv \exists x \forall y \neg ((x>y) \vee (y>x^2)) \\
&\equiv \exists x \forall y \ (\neg (x>y) \wedge \neg (y>x^2)) \\
&\equiv \exists x \forall y \ ((x \leq y) \wedge (y \leq x^2))
\end{aligned}$$

The original statement is True.
Explanation: In the original statement, given an arbitrary x, let y=x-1. Then x>y is always true. $((x>y) \vee (y>x^2))$ is always true too. Therefore the original statement is True.

5. (4 points) Prove that $\sqrt[3]{2}$ is irrational.

Note: This question is similar to an example in class, which is to prove $\sqrt{2}$ is irrational.

Proof by contradiction:
Assume $\sqrt[3]{2}$ is rational.
By definition, it can be written as $\sqrt[3]{2} = a/b$, where a and b are two integers without common factors (except 1). (*)
Then $2 = a^3/b^3$
$2b^3 = a^3$
Then a^3 is even.
We can prove that a is even too.

{Proof by contrapositive (if a^3 is even, then a is even): Assume a is odd, then a=2n+1 for some integer n. $a^3 = (2\text{n} + 1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1$.
a^3 is odd, which is the negation of $a^3$is even. Q.E.D.}
By definition, a=2k for some integer k.
$2b^3 = a^3$=$(2k)^3 = 8k^3$
$b^3 = 4k^3 = 2(2k^3)$
Then b^3 is an even integer.
Then b is an even integer.
Now a and b are both even, which means a and b have a common factor 2. It is contradict to (*).
Therefore, the assumption is wrong, and $\sqrt[3]{2}$ is irrational.

6. (5 points) Show that for any given 8 integers, there exist two of them whose difference is divisible by 7.

Suppose we have 8 integers $a_1, a_2, \dots, a_8$.
When dividing an integer by 7, the quotient could only be one of the following numbers: 0, 1, 2, 3, 4, 5, 6.
There are totally 7 possibilities for the quotient.
$\therefore At\ least\ two\ integers\ in\ a_1, a_2, \dots, a_8$ will have the same quotient.
{Proof by contradiction: Assume none of the integers in $a_1, a_2, \dots, a_8$ will have the same quotient, then there will be at least 8 possibilities, which is contradict to 7 possibilities in total. Q.E.D. }
If two integers have the same quotient, that means these two integers can be written in the following format: 7m+k and 7n+k. Where k is the quotient, and m and n are two integers.
Their difference (7m+k)- (7n+k)=7(m-n) is divisible by 7.
Q.E.D.

<u>Note: Recall the example in class: “If n+1 balls are distributed among n bins, prove that at least one bin has more than 1 ball”.</u>